

Corrigendum for the comparison theorems in
 ”A new definition of viscosity solutions for a
 class of second-order degenerate elliptic
 integro-differential equations”.

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In this note, we shall present the correction of the proofs of the comparison results in the paper [1]. In order to show clearly the correct way of the demonstration, we shall simplify the problem to the following.

$$\begin{aligned} \text{(Problem (I)) : } \quad & F(x, u, \nabla u, \nabla^2 u) - \int_{\mathbf{R}^N} u(x+z) - u(x) \\ & - \mathbf{1}_{|z| \leq 1} \langle z, \nabla u(x) \rangle q(dz) = 0 \quad \text{in } \Omega, \end{aligned} \quad (1)$$

$$\begin{aligned} \text{(Problem (II)) : } \quad & F(x, u, \nabla u, \nabla^2 u) - \int_{\{z \in \mathbf{R}^N \mid x+z \in \bar{\Omega}\}} u(x+z) - u(x) \\ & - \mathbf{1}_{|z| \leq 1} \langle z, \nabla u(x) \rangle q(dz) = 0 \quad \text{in } \Omega, \end{aligned} \quad (2)$$

where $\Omega \subset \mathbf{R}^N$ is open, and $q(dz)$ is a positive Radon measure such that $\int_{|z| \leq 1} |z|^2 q(dz) + \int_{|z| > 1} 1 q(dz) < \infty$. Although in [1] only (II) was studied, in order to avoid the non-essential technical complexity, here, let us give the explanation mainly for (I). For (I), we consider the Dirichlet B.C.:

$$u(x) = g(x) \quad \forall x \in \Omega^c, \quad (3)$$

where g is a given continuous function in Ω^c . For (II), we assume that Ω is a precompact convex open subset in \mathbf{R}^N with C^1 boundary satisfying the uniform exterior sphere condition, and consider either the Dirichlet B.C.:

$$u(x) = h(x) \quad \forall x \in \partial\Omega, \quad (4)$$

where h is a given continuous function on $\partial\Omega$, or the Neumann B.C.:

$$\langle \nabla u(x), \mathbf{n}(x) \rangle = 0 \quad \forall x \in \partial\Omega, \quad (5)$$

where $\mathbf{n}(x) \in \mathbf{R}^N$ the outward unit normal vector field defined on $\partial\Omega$. The above problems are studied in the framework of the viscosity solutions introduced in [1]. Under all the assumptions in [1], for (I) the following comparison result holds, and for (II), although the proofs therein are incomplete, the comparison results stated in [1] hold, and we shall show in a future article.

Theorem 1.1 (Problem I with Dirichlet B.C.) *Assume that Ω is bounded, and the conditions for F in [1] hold. Let $u \in USC(\mathbf{R}^N)$ and $v \in LSC(\mathbf{R}^N)$ be respectively a viscosity subsolution and a supersolution of (1) in Ω , which satisfy $u \leq v$ on Ω^c . Then, $u \leq v$ in Ω .*

To prove Theorem 1.1, we approximate the solutions u and v by the sup-convolution: $u^r(x) = \sup_{y \in \mathbf{R}^N} \{u(y) - \frac{1}{2r^2}|x - y|^2\}$ and the infconvolution: $v_r(x) = \inf_{y \in \mathbf{R}^N} \{v(y) + \frac{1}{2r^2}|x - y|^2\}$ ($x \in \mathbf{R}^N$), where $r > 0$.

Lemma 1.2 (Approximation for Problem (I)) *Let u and v be respectively a viscosity subsolution and a supersolution of (1). For any $\nu > 0$ there exists $r > 0$ such that u^r and v_r are respectively a subsolution and a supersolution of the following problems.*

$$F(x, u, \nabla u, \nabla^2 u) - \int_{\mathbf{R}^N} u(x + z) - u(x) - \mathbf{1}_{|z| \leq 1} \langle z, \nabla u(x) \rangle q(dz) \leq \nu, \quad (6)$$

$$F(x, v, \nabla v, \nabla^2 v) - \int_{\mathbf{R}^N} v(x + z) - v(x) - \mathbf{1}_{|z| \leq 1} \langle z, \nabla v(x) \rangle q(dz) \geq -\nu, \quad (7)$$

in $\Omega_r = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \sqrt{2Mr}\}$, where $M = \max\{\sup_{\overline{\Omega}} |u|, \sup_{\overline{\Omega}} |v|\}$.

Remark that u^r is semiconvex, v_r is semiconcave, and both are Lipschitz continuous in \mathbf{R}^N . We then deduce from the Jensen's maximum principle and the Alexandrov's theorem (deep results in the convex analysis, see [2] and [3]), the following lemma, the last claim of which is quite important in the limit procedure in the nonlocal term.

Lemma 1.3 *Let U be semiconvex and V be semiconcave in Ω . For $\phi(x, y) = \alpha|x - y|^2$ ($\alpha > 0$) consider $\Phi(x, y) = U(x) - V(y) - \phi(x, y)$, and assume that (\bar{x}, \bar{y}) is an interior maximum of Φ in $\bar{\Omega} \times \bar{\Omega}$. Assume also that there is an open precompact subset O of $\Omega \times \Omega$ containing (\bar{x}, \bar{y}) , and that $\mu = \sup_O \Phi(x, y) - \sup_{\partial O} \Phi(x, y) > 0$. Then, the following holds.*

- (i) *There exists a sequence of points $(x_m, y_m) \in O$ ($m \in \mathbf{N}$) such that $\lim_{m \rightarrow \infty} (x_m, y_m) = (\bar{x}, \bar{y})$, and $(p_m, X_m) \in J_{\Omega}^{2,+} U(x_m)$, $(p'_m, Y_m) \in J_{\Omega}^{2,-} V(y_m)$ such that $\lim_{m \rightarrow \infty} p_m = \lim_{m \rightarrow \infty} p'_m = 2\alpha(x_m - y_m) = p$, and $X_m \leq Y_m \quad \forall m$.*
- (ii) *For $P_m = (p_m - p, -(p'_m - p))$, $\Phi_m(x, y) = \Phi(x, y) - \langle P_m, (x, y) \rangle$ takes a maximum at (x_m, y_m) in O .*
- (iii) *The following holds for any $z \in \mathbf{R}^N$ such that $(x_m + z, y_m + z) \in O$.*

$$U(x_m + z) - U(x_m) - \langle p_m, z \rangle \leq V(y_m + z) - V(y_m) - \langle p'_m, z \rangle. \quad (8)$$

By admitting these lemmas here, let us show how Theorem 1.1 is proved. *Proof of Theorem 1.1.* We use the argument by contradiction, and assume that $\max_{\bar{\Omega}}(u - v) = (u - v)(x_0) = M_0 > 0$ for $x_0 \in \Omega$. Then, we approximate u by u^r (supconvolution) and v by v_r (infconvolution), which are a subsolution and a supersolution of (6) and (7), respectively. Clearly, $\max_{\bar{\Omega}}(u^r - v_r) \geq M_0 > 0$. Let $\bar{x} \in \Omega$ be the maximizer of $u^r - v_r$. In the following, we abbreviate the index and write $u = u^r$, $v = v_r$ without any confusion. As in the PDE theory, consider $\Phi(x, y) = u(x) - v(y) - \alpha|x - y|^2$, and let (\hat{x}, \hat{y}) be the maximizer of Φ . Then, from Lemma 1.3 there exists $(x_m, y_m) \in \Omega$ ($m \in \mathbf{N}$) such that $\lim_{m \rightarrow \infty} (x_m, y_m) = (\hat{x}, \hat{y})$, and we can take $(\varepsilon_m, \delta_m)$ a pair of positive numbers such that $u(x_m + z) \leq u(x_m) + \langle p_m, z \rangle + \frac{1}{2} \langle X_m z, z \rangle + \delta_m |z|^2$, $v(y_m + z) \geq v(y_m) + \langle p'_m, z \rangle + \frac{1}{2} \langle Y_m z, z \rangle - \delta_m |z|^2$, for $\forall |z| \leq \varepsilon_m$. From the definition of the viscosity solutions, we have

$$\begin{aligned} & F(x_m, u(x_m), p_m, X_m) - \int_{|z| \leq \varepsilon_m} \frac{1}{2} \langle (X_m + 2\delta_m I)z, z \rangle dq(z) \\ & - \int_{|z| \geq \varepsilon_m} u(x_m + z) - u(x_m) - \mathbf{1}_{|z| \leq 1} \langle z, p_m \rangle q(dz) \leq \nu, \end{aligned}$$

$$\begin{aligned}
& F(y_m, v(y_m), p'_m, Y_m) - \int_{|z| \leq \varepsilon_m} \frac{1}{2} \langle (Y_m - 2\delta_m I)z, z \rangle dq(z) \\
& - \int_{|z| \geq \varepsilon_m} v(y_m + z) - v(y_m) - \mathbf{1}_{|z| \leq 1} \langle z, p'_m \rangle q(dz) \geq -\nu.
\end{aligned}$$

By taking the difference of the above two inequalities, by using (8), and by passing $m \rightarrow \infty$ (thanking to (8), it is now available), we can obtain the desired contradiction. The claim $u \leq v$ is proved.

Remark 1.1. To prove the comparison results for (II) (in [1]), we do the approximation by the supconvolution: $u^r(x) = \sup_{y \in \bar{\Omega}} \{u(y) - \frac{1}{2r^2}|x - y|^2\}$, and the infconvolution: $v_r(x) = \inf_{y \in \bar{\Omega}} \{v(y) + \frac{1}{2r^2}|x - y|^2\}$ as in Lemma 1.2. Because of the restriction of the domain of the integral of the nonlocal term and the Neumann B.C., a slight technical complexity is added. The approximating problem for (2)-(5) in $\bar{\Omega}$ is as follows.

$$\begin{aligned}
& \min[F(x, u(x), \nabla u(x), \nabla^2 u(x)) + \min_{y \in \bar{\Omega}, |x-y| \leq \sqrt{2Mr}} \{ - \int_{\{z \in \mathbf{R}^N | \mathbf{y} + \mathbf{z} \in \bar{\Omega}\}} u(x+z) - u(x) \\
& - \mathbf{1}_{|z| \leq 1} \langle z, \nabla u(x) \rangle q(dz), \min_{y \in \partial\Omega, |x-y| \leq \sqrt{2Mr}} \{ \langle \mathbf{n}(y), \nabla u(x) \rangle + \rho \}] \leq \nu, \\
& \max[F(x, v(x), \nabla v(x), \nabla^2 v(x)) + \max_{y \in \bar{\Omega}, |x-y| \leq \sqrt{2Mr}} \{ - \int_{\{z \in \mathbf{R}^N | \mathbf{y} + \mathbf{z} \in \bar{\Omega}\}} v(x+z) - v(x) \\
& - \mathbf{1}_{|z| \leq 1} \langle z, \nabla v(x) \rangle q(dz), \max_{y \in \partial\Omega, |x-y| \leq \sqrt{2Mr}} \{ \langle \mathbf{n}(y), \nabla v(x) \rangle - \rho \}] \geq -\nu.
\end{aligned}$$

We deduce the comparison result from this approximation and Lemma 1.3, by using the similar argument as in the proof of Theorem 1.1.

References

- [1] M. Arisawa, A new definition of viscosity solutions for a class of second-order degenerate elliptic integro-differential equations.
- [2] M.G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations. Bulletin of the AMS, vol.27, no. 1 1992.
- [3] W.H. Fleming and H.M. Soner, Controlled Markov processes and Viscosity solutions, Springer-Verlag 1992.